CHURCHLANDS SENIOR HIGH SCHOOL
MATHEMATICS SPECIALIST 3, 4 TEST ONE 2017
NON-Calculator Section
Chapters 1, 2,

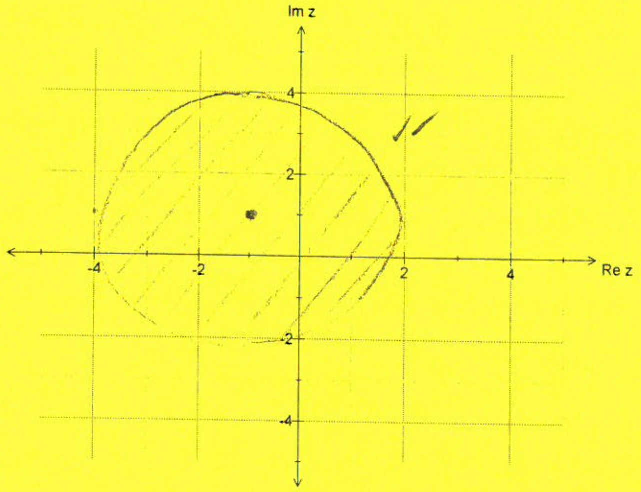
Name________________________________ **Time: 50 minutes**
Total: 49 marks

1.[12 marks: 3,3,3,3]
Describe and sketch each of the following subsets of the complex plane.

a) $\{z : |z + 1 - i| \leq 3\}$

$|z + 1 - i| \leq 3$

$|z - (-1 + i)| \leq 3$



Description : Circular region centre $(-1, i)$ with radius ≤ 3

b) $\{z : |z + 2 - i| = |z - 1 + 2i|\}$

$|z - (-2 + i)| = |z - (1 - 2i)|$

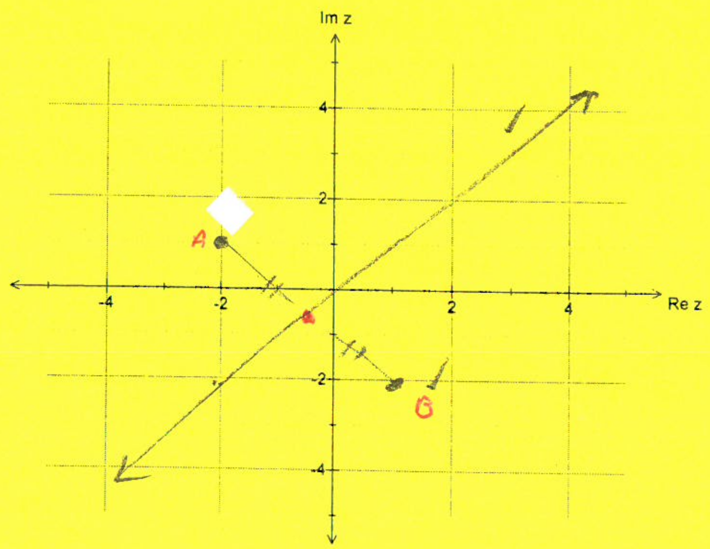
$(-2, 1)\ (1, -2)$

$m = \frac{-3}{3}$

$= -1.$

midpoint of AB $= \left(\frac{-2+1}{2}, \frac{1-2}{2}\right)$

$= \left(-\frac{1}{2}, -\frac{1}{2}\right)$



Description: The line with equation $y = x$

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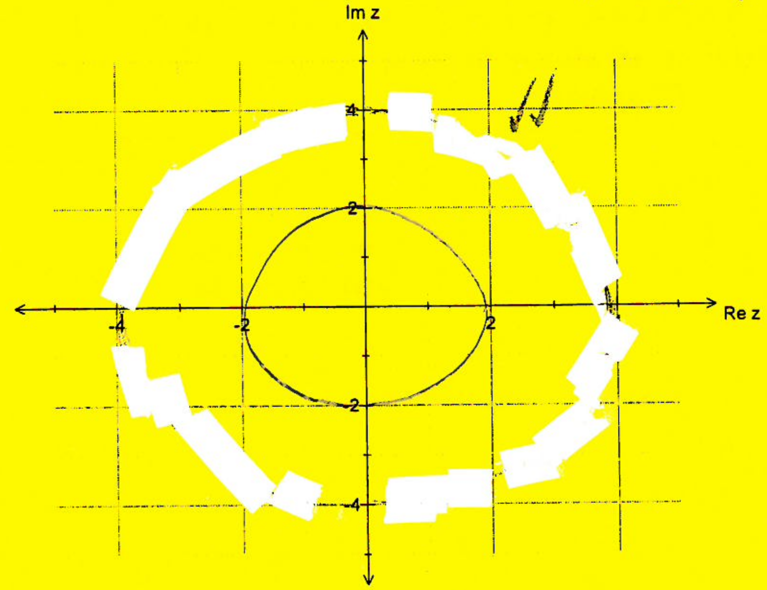
c) $\{z: z\bar{z} = 4\}$

Let $z = x + yi$

$\therefore z\bar{z} = 4 \Rightarrow (x+yi)(x-yi) = 4$

$x^2 + y^2 = 4$



Description: Circle centre $(0,0)$ radius 2. ✓

d) $\{z: Re z - Im z \leq 9\}$

$Re\, z - Im\, z \leq 9$

$\Rightarrow x - y \leq 9$

Description:

The half planar region above and including the line with equation $x - y = 9$ ✓

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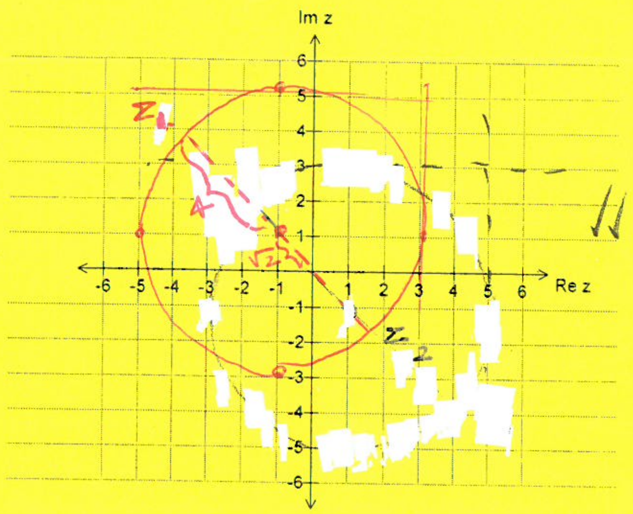
2. [6 marks: 2,1,1,1,1]

Sketch on the complex plane below the region defined by

$|z - i + 1| = 4$

$|z - (-1 + i)| = 4$



Hence, find exactly.

i) the maximum value of $|z|$ Maximum value of $|z_1|$ is $4 + \sqrt{2}$ units ✓

ii) the minimum value of $|z|$ Minimum value of $|z_2|$ is $4 - \sqrt{2}$ units ✓

iii) the maximum value of Re(z) Max value of Re(z) is 3 ✓

iv) the minimum value of Im(z) Min value of Im(z) is −3 ✓

3. [7marks: 2,1,4]

i) Find the remainder when $x^3 - 4x^2 + 7x - 6$ is divided by $x - 2$.

Let $f(x) = x^3 - 4x^2 + 7x - 6$

Remainder is $f(2) = 2^3 - 4(2)^2 + 7(2) - 6$

$= 8 - 16 + 14 - 6$

$= 0$ ✓

ii) When $x^3 - x^2 + cx - 3$ is divided by $x - 3$, the remainder is 30. Find c.

Let $f(x) = x^3 - x^2 + cx - 3$

$\Rightarrow f(3) = 30$ ✓

ie $3^3 - 3^2 + c(3) - 3 = 30$

$27 - 9 + 3c - 3 = 30$

$3c = 15$

$\therefore c = 5$ ✓

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iii) When $3x^3 - ax^2 - bx + 1$ is divided by $x - 2$, the remainder is 15. If $x - 1$ is a factor of the given polynomial, find the values of a and b.

Let $f(x) = 3x^3 - ax^2 - bx + 1$

$f(2) = 15$

$\Rightarrow \; 24 - 4a - 2b + 1 = 15$

ie $\; -4a - 2b = -10$

ie $\; 4a + 2b = 10$

$2a + b = 5 \quad \text{—— (1)}$ ✓

Also $f(1) = 0$

$3 - a - b + 1 = 0$

$a + b = 4 \quad \text{—— (2)}$ ✓

$2a + b = 5$

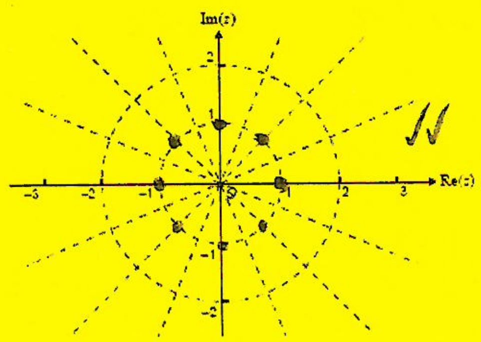
$2a - a + 4 = 5$

$a = 1$

$\therefore a = 1$ ✓

Thus $b = 3$ ✓

4. [2 marks]
Plot the roots of $z^8 = 1$ on the Argand diagram below.



✓✓

$z = 1$ is one such root ie $(1 + 0i)$
all other 7 roots are equally spaced around the circle

5. [4 marks:1,2,1]

Let $\beta = 1 - i\sqrt{3}$.
i) Express β in polar form.

$r = \sqrt{1^2 + (-\sqrt{3})^2}$

$= \sqrt{1 + 3}$

$= 2$

$\tan\theta = \frac{-3}{1}$

$\theta = -\frac{\pi}{3}$

4TH Quad

$\therefore \beta = 2\,\mathrm{cis}\left(-\frac{\pi}{3}\right)$ ✓

ii) Express β^5 in polar form.

$\beta^5 = \left[2\,\mathrm{cis}\left(-\frac{\pi}{3}\right)\right]^5$

$= 2^5\,\mathrm{cis}\left(-\frac{5\pi}{3}\right)$ ✓

$= 32\,\mathrm{cis}\left(\frac{\pi}{3}\right)$ ✓

iii) Hence express β^5 in the form $x + iy$.

$\beta^5 = 32\left[\cos\frac{\pi}{3} + \sin\left(\frac{\pi}{3}\right)i\right]$

$= 32\left[\frac{1}{2} + \frac{\sqrt{3}}{2}i\right]$

$= 16 + 16\sqrt{3}i$ ✓

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6. [12 marks:3, 3, 6 marks]

(a) If $z_1 = 3cis\left(\frac{4\pi}{3}\right)$ and $z_2 = \frac{1}{2}cis\left(\frac{\pi}{6}\right)$, prove that: $\frac{z_1}{z_2} = -3(\sqrt{3}+i)$

$$\frac{z_1}{z_2} = \frac{3\,cis\left(\frac{4\pi}{3}\right)}{\frac{1}{2}\,cis\left(\frac{\pi}{6}\right)}$$

$$= \frac{6\,cis\left(\frac{4\pi}{3}\right)}{cis\left(\frac{\pi}{6}\right)} \checkmark$$

$$= 6\,cis\left(\frac{4\pi}{3} - \frac{\pi}{6}\right)$$

$$= 6\,cis\left(\frac{7\pi}{6}\right)$$

$$= 6\,cis\left(-\frac{5\pi}{6}\right) \checkmark$$

$$6\,cis\left(-\frac{5\pi}{6}\right)$$

$$= 6\left[\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right]$$

$$= 6\left[-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right]$$

$$= \frac{6}{2}\left[-\sqrt{3} - i\right]$$

$$= 3\left[-\sqrt{3} - i\right]$$

$$= -3\left[\sqrt{3} + i\right] \checkmark$$

which was to be shown

b) Simplify $\frac{\left(3cis\frac{\pi}{3}\right)\left(4cis\frac{\pi}{2}\right)}{6cis\frac{\pi}{4}}$ giving your answer in the form $rcis\theta$.

$$\frac{3\,cis\left(\frac{\pi}{3}\right)4\,cis\left(\frac{\pi}{2}\right)}{6\,cis\left(\frac{\pi}{4}\right)}$$

$$= \frac{12\,cis\left(\frac{\pi}{3} + \frac{\pi}{2}\right)}{6\,cis\left(\frac{\pi}{4}\right)} \checkmark$$

$$= \frac{2\,cis\left(\frac{5\pi}{6}\right)}{cis\left(\frac{\pi}{4}\right)} \checkmark$$

$$= 2\,cis\left(\frac{5\pi}{6} - \frac{\pi}{4}\right)$$

$$= 2\,cis\left(\frac{10\pi}{12} - \frac{3\pi}{12}\right)$$

$$= 2\,cis\left(\frac{7\pi}{12}\right) \checkmark$$

c)
Find all $z = x + iy$, given that $z\bar{z} + 3z = \bar{z} + 4i$

$(x+iy)(x-iy)+3(x+iy) = x-yi+4i$

$x^2+y^2+3x+3iy = x+(4-y)i$

$Re(LHS) = x^2+y^2+3x$
$Im(LHS) = 3y$
$Re(RHS) = x$
$Im(RHS) = (4-y)$

$\therefore \begin{cases} x^2+y^2+3x = x \\ 3y = 4-y \end{cases}$

ie $\begin{cases} x^2+y^2+2x = 0 \\ 4y = 4 \end{cases}$

Thus $y = 1$
and $x^2+1+2x = 0$
$x^2+2x+1 = 0$
$(x+1)(x+1) = 0$
$\Rightarrow x = -1$.

Hence $z = -1+i$

7. [6 marks]
If $z = cis\theta$ and by using De Moivre's theorem together with knowledge of the binomial expansion to find z^3, show that $cos3\theta = cos^3\theta - 3cos\theta sin^2\theta$ and $sin3\theta = 3cos^2\theta sin\theta - sin^3\theta$.

$z^3 = (cis\theta)^3$

$= cis3\theta$ using De Moivre's theorem

$= cos3\theta + i sin3\theta$ —— (1)

But $z = [cos\theta + i sin\theta]$

$\therefore z^3 = [cos\theta + i sin\theta]^3$

using binomial expansion gives

$1 cos^3\theta + 3cos^2\theta(i sin\theta) + 3cos\theta(i sin\theta)^2 + (i sin\theta)^3$

$= cos^3\theta + 3 sin\theta cos^2\theta i - 3cos\theta sin^2\theta - sin^3\theta i$

$= cos^3\theta - 3cos\theta sin^2\theta + [3 sin\theta cos^2\theta - sin^3\theta] i$

Equating real & imaginary parts in (1)

gives $cos3\theta = cos^3\theta - 3cos\theta sin^2\theta$

and $sin3\theta = 3cos^2\theta sin\theta - sin^3\theta$ which was to be shown

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